Government College of Engineering and Research
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Fundamental of Finite Element Analysis

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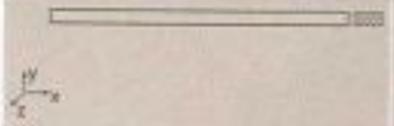
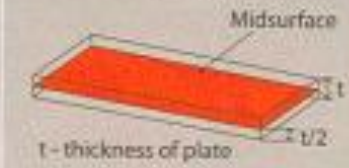
Unit 3
2D Elements

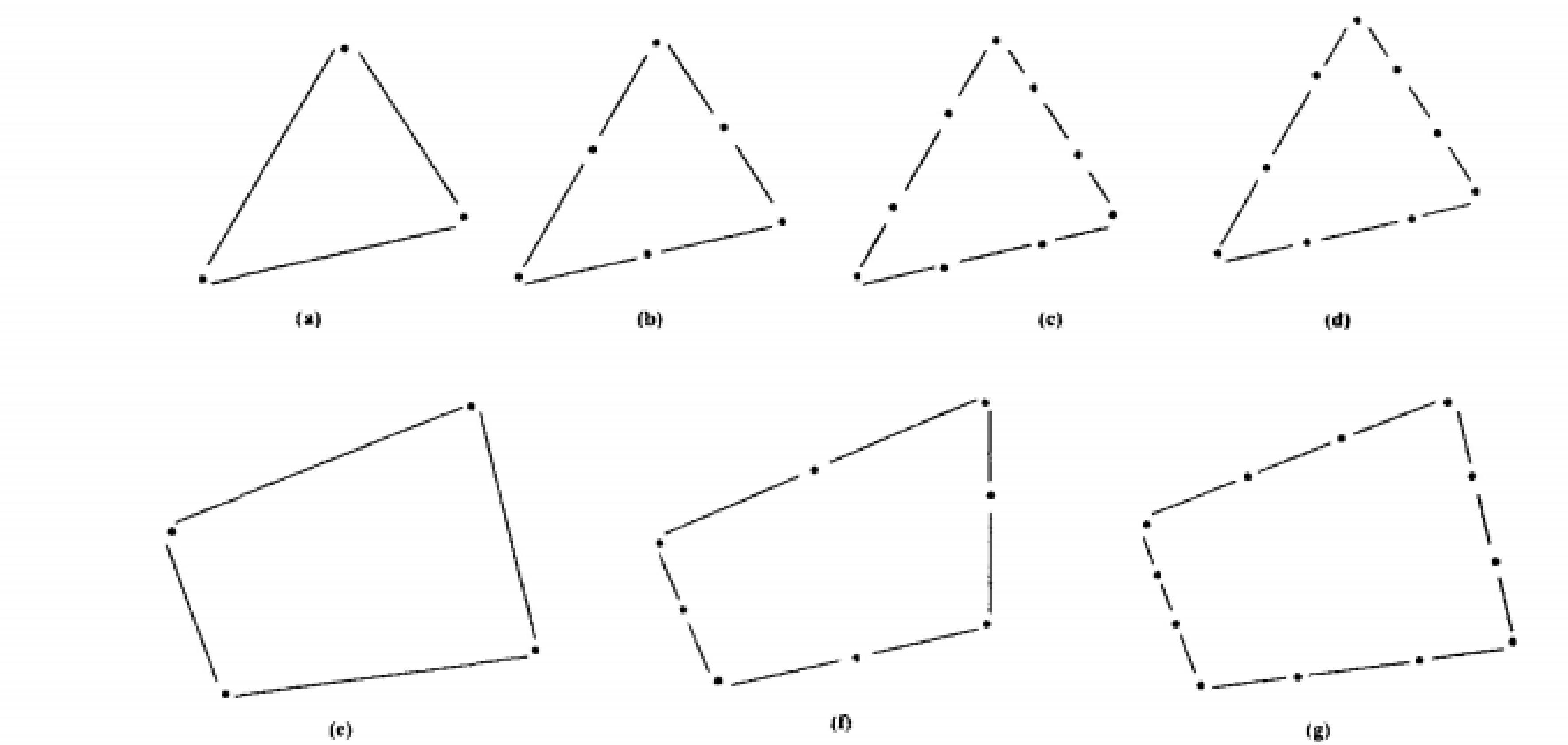
Type of Element

Elements

1-d	2-d	3-d	Other
 x >>> y, z	 x , z >>> y	 x ~ y ~ z	
One of the dimension is very large in comparison to rest of the two Element shape – line. Additional data from user - remaining two dimensions i.e. area of c/s Element type – rod, bar, beam, pipe, axi-symmetric shell etc Practical applications - Long shafts, beams, pin joint, connection elements etc.	Two of the dimensions are very large in comparison to third one Element shape – quad, tria Additional data from user - remaining dimension i.e. thickness Element type – thin shell, plate, membrane, plane stress, plane strain, , axi-symmetric solid etc. Practical applications - Sheet metal parts, plastic components like instrument panel etc.	All dimensions are comparable Element shape – tetra, penta, hex, pyramid Additional data from user – none Element type – solid Practical applications - Transmission casing, engine block, crankshaft etc.	Mass – Pt. element, concentrated mass at C.G. of the component Spring – translational & rotational stiffness Damper - damping coefficient Gap – Gap distance, stiffness, friction Rigid – RBE2, RBE3 Weld

Type of 2D Elements

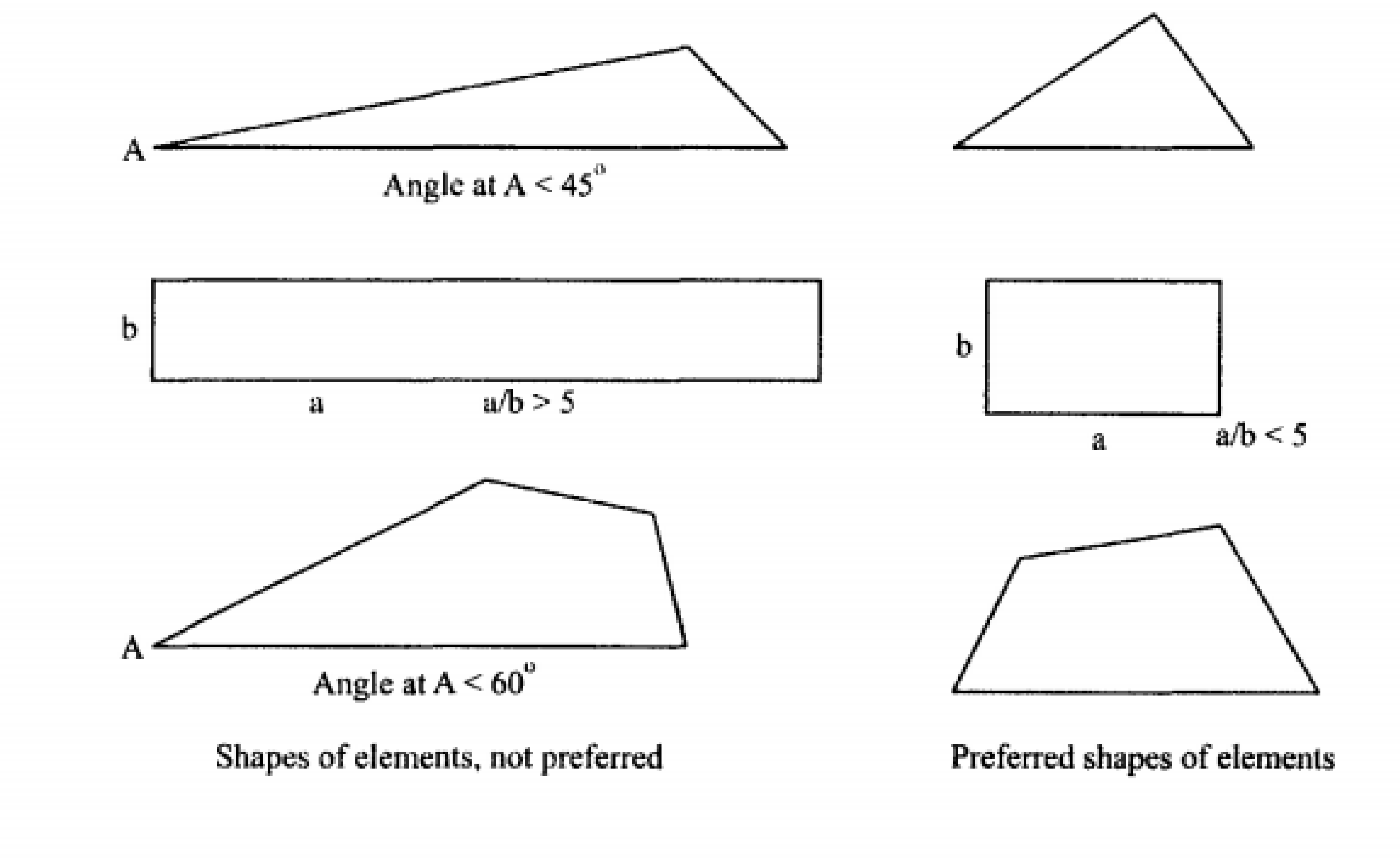
This chapter considers the two-dimensional finite element. Two-dimensional (planar) elements are defined by three or more nodes in a two-dimensional plane



2D Elements

Element	Order of Displacement	No.of nodes	Terms included	Polynomial type
Triangle (Fig.5.2 a)	Linear	3	$a_1 + a_2.x + a_3.y$	Complete & Isotropic
Triangle (Fig.5.2 b)	Quadratic	6	$a_1 + a_2\ x + a_3\ y + a_4.x^2 + a_5\ xy + a_6\ y^2$	Complete & Isotropic
Triangle (Fig.5.2 c)	Cubic	9	$a_1 + a_2.x + a_3.y + a_4.x^2 + a_5.xy + a_6.y^2 + a_7.x^2y + a_8.xy^2 + a_9\ x^2y^2$ $a_1 + a_2x + a_1y + a_4x^2 + a_5xy + a_6y^2 + a_7x^2y + a_8xy^2 + a_9x^3$ (Not preferred)	Incomplete, Isotropic Incomplete Non-Isotropic
Triangle (Fig.5.2 d)	Cubic	10	$a_1 + a_2.x + a_3.y + a_4.x^2 + a_5.xy + a_6.y^2 + a_7.x^3 + a_8.x^2y + a_9\ xy^2 + a_{10}.y^3$	Complete, Isotropic
Quadrilateral (Fig.5.2 e)	Linear	4	$a_1 + a_2.x + a_3\ y + a_4.xy$ $a_1 + a_2x + a_3y + a_4x^2$ (Not preferred)	Incomplete, Isotropic Incomplete, Non-Isotropic
Quadrilateral (Fig.5.2 f)	Quadratic	8	$a_1 + a_2.x + a_3.y + a_4.x^2 + a_5.xy + a_6.y^2 + a_7.x^2y + a_8.xy^2$	Incomplete, Isotropic
Quadrilateral (Fig.5.2 g)	Cubic	12	$a_1 + a_2.x + a_3.y + a_4.x^2 + a_5.xy + a_6.y^2 + a_7.x^3 + a_8.x^2y + a_9.xy^2 + a_{10}.y^3 + a_{11}.x^3y + a_{12}.xy^3$	Incomplete, Isotropic

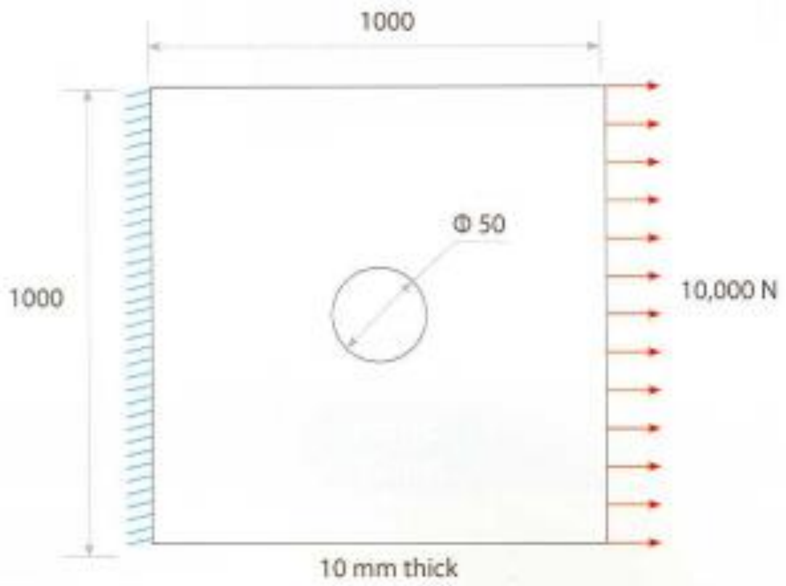
Aspect Ration in 2D element



Case study & Practical Assignment

Comparison of Triangular and Quadrilateral elements:

We will carry out plate with circular hole exercise to compare performance of different elements with known analytical answer.



Analytical answer : max. stress = 3 N/mm²

For Infinite plate with very small circular hole, Stress Concentration Factor (SCF) =3,

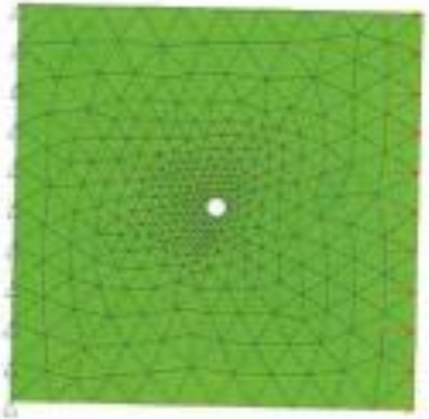
SCF = max. stress / nominal stress

Nominal stress = F/A = 10,000 /(1000*10) =1 N/mm²

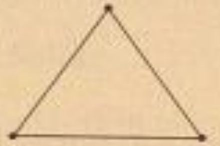
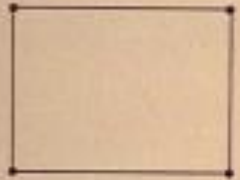
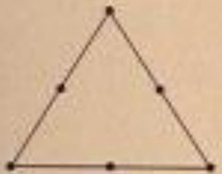
3 = max. stress / 1

max. stress = 3 N/mm²

Exact Answer : 3 N/mm²

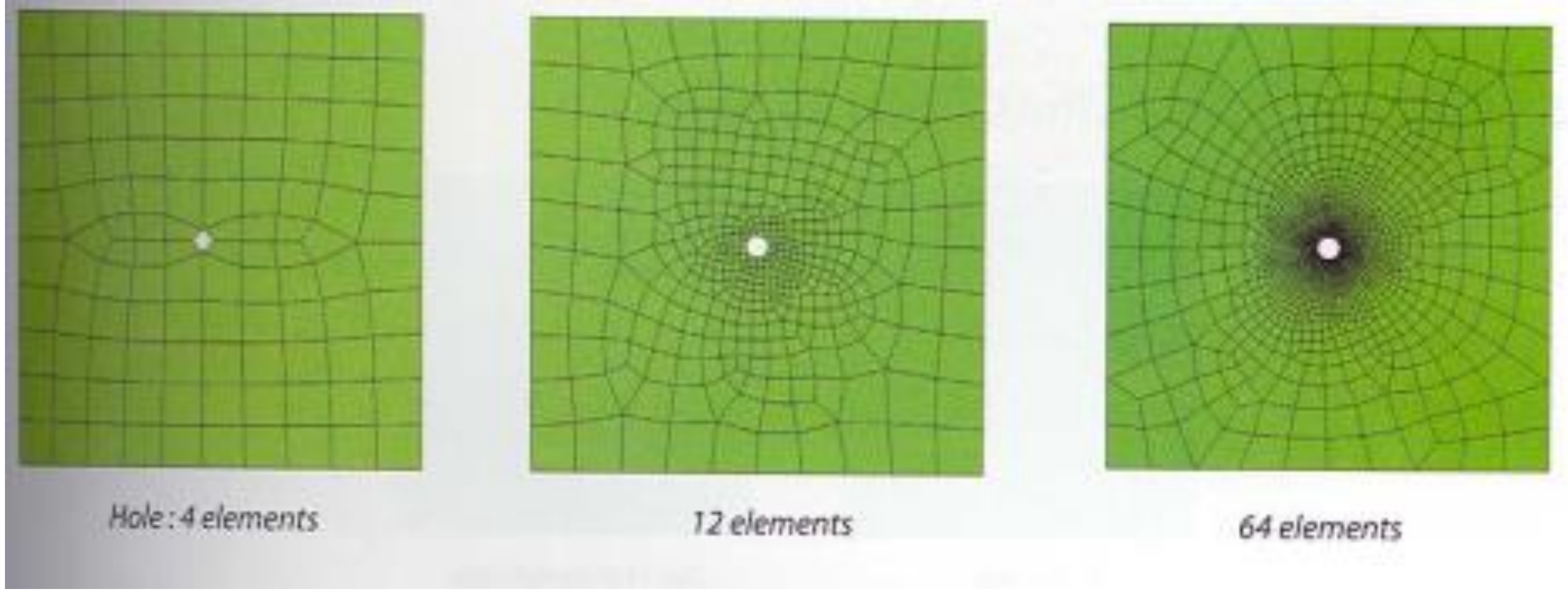


Boundary condition plot for Tria 3 model

Type of Element	Stress N/mm²	Displacement function
Linear Tria 3  CST (Constant Strain Triangle)	1.70	$u = a_0+a_1x+a_2y$ (3 nodes – 3 terms in displacement function) Strain = $\varepsilon_x = \frac{\partial u}{\partial x} = a_1 = \text{const.}$ $\varepsilon_y = \frac{\partial u}{\partial y} = a_2 = \text{const.}$
Linear Quad 4 	2.20	$u = a_0+a_1x+a_2y+a_3xy$ (one additional term in comparison to tria 3, makes it more accurate)
Parabolic Tria 6  LST (Linear Strain Triangle)	2.75	$u = a_0+a_1x+a_2y+a_3x^2+a_4y^2 +a_5xy$ (6 nodes – 6 terms in displacement function)
Parabolic Quad 8 	2.94	$u = a_0+a_1x+a_2y+a_3xy+a_4x^2 +a_5y^2 +a_6x^2y +a_7xy^2$ (two additional terms in comparison to tria 6, makes it more accurate)

Conclusion :

- *Quad elements are better than triangular elements.*
- *Parabolic elements are better than linear elements.*



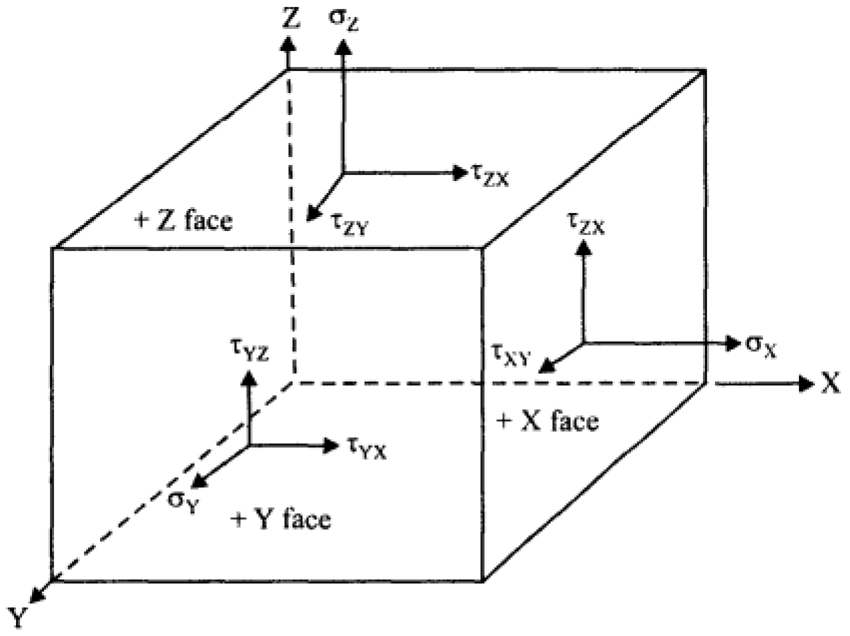
Exact Answer : 3 N/mm²

No. elements on hole	Stress, N/mm²	Displacement	Nodes	Elements
4	1.23	0.0048	136	114
6	1.77	0.0048	277	254
8	2.20	0.0048	369	345
12	2.65	0.0048	428	402
16	2.78	0.0048	493	465
32	2.92	0.0048	1161	1125
64	3.02	0.0048	2530	2478

Conclusion:

- *More the number of elements in critical region better is the accuracy*

Introduction of stress tensor



$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Normal stresses on the diagonal
Shear stresses off diagaonal
$\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

The normal and shear stresses on a stress element in 3D can be assembled into a 3x3 matrix known as the **stress tensor**.

Distortion Energy Theory or Von-Misses Theory - Ductile Material

- **Equivalent stress is**

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

For 2D and 3D problem

Hooke's Law for 3D stress

- **Hooke's Law can also be applied to material undergoing three dimensional stress (triaxial loading).**
- **The development of 3D equations is similar to 1D, sum the total normal strain in one direction due to loads in all three directions. For the x-direction, this gives,**

$$\varepsilon_{x\ \text{total}} = \varepsilon_{x\ \text{due to}\ \sigma x} + \varepsilon_{x\ \text{due to}\ \sigma y} + \varepsilon_{x\ \text{due to}\ \sigma z}$$

$$= \sigma_x /E - \nu\sigma_y/E - \nu\sigma_z/E$$

$$\varepsilon_x = (\sigma_x - \nu\sigma_y - \nu\sigma_y) / E$$

Similarly, the other directions can also be determined. The final equations are summarized in the table below.

3D Hooke's Law (Stress-Strain Relationship)		
Compliance Format		Stiffness Format
$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$		$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$
$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$		$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\right]$
$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$		$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$
$\gamma_{xy} = \frac{\tau_{xy}}{G};\ \gamma_{yz} = \frac{\tau_{yz}}{G};\ \gamma_{xz} = \frac{\tau_{xz}}{G}$		$\tau_{xy} = G\gamma_{xy};\ \tau_{yz} = G\gamma_{yz};\ \tau_{xz} = G\gamma_{xz}$

The shear modulus is related to Young modulus and Poisson's ratio,

$G = E / 2(1 + \nu)$

Matrix form stain and stress

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

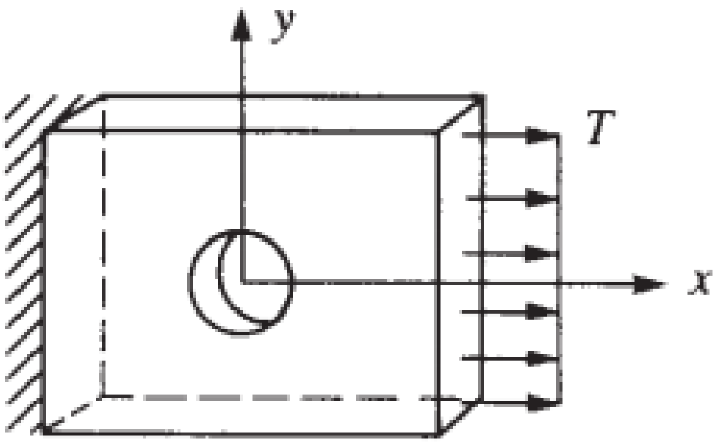
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

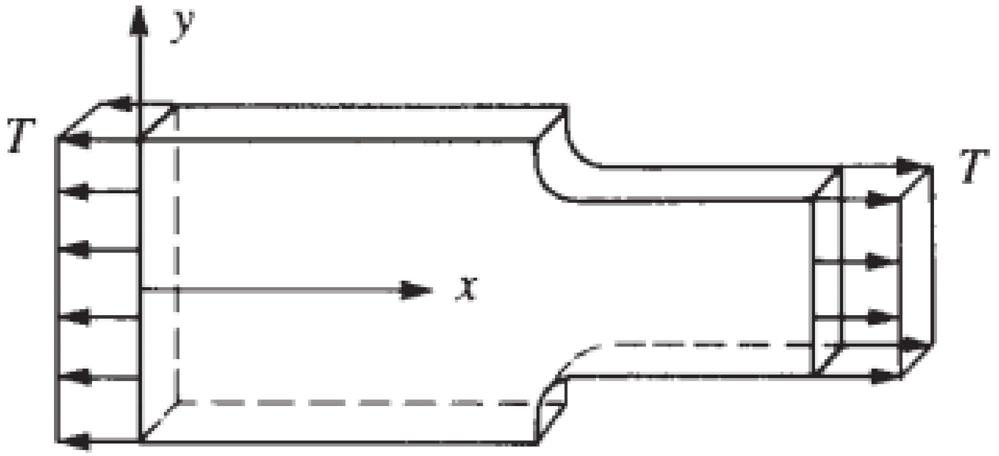
Two dimensional stress-strain relationships are summarized in the table below.

2D Hooke's Law (Stress-Strain Relationship)		
Compliance Format		**Stiffness Format**
$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$ $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$ $\gamma = \frac{\tau}{G} = \frac{1}{E}[2(1+\nu)\tau]$		$\sigma_x = \frac{(\varepsilon_x + \nu\varepsilon_y)E}{(1-\nu^2)}$ $\sigma_y = \frac{(\varepsilon_y + \nu\varepsilon_x)E}{(1-\nu^2)}$ $\tau = G\gamma = \frac{E\gamma}{2(1+\nu)}$
or in matrix form $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix}$		or in matrix form $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & 0 \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix}$

- **Plane Stress**

Plane stress is defined to be a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero





- **Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the x-y plane can be considered to be under plane stress.**

- **The plates in the x-y plane shown subjected to surface tractions T (pressure acting on the surface edge or face of a member in units of force/area) in the plane are under a state of plane stress; that is, the normal stress Ϭz and the shear stresses τxz and τyz are assumed to be zero.**

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

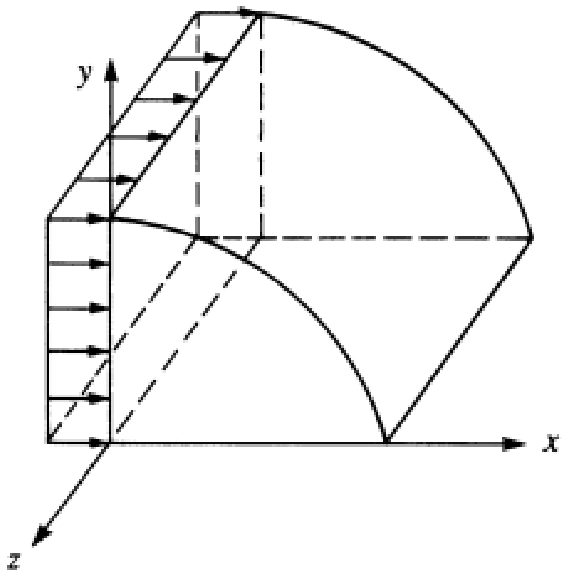
$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & 0 \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

- **Plane Strain**

Plane strain is defined to be a state of strain in which the strain normal to the x-y plane εz and the shear strains γxz and γ yz are assumed to be zero



Dam subjected to horizontal loading;

The assumptions of plane strain are realistic for long bodies (say, in the z direction) with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.

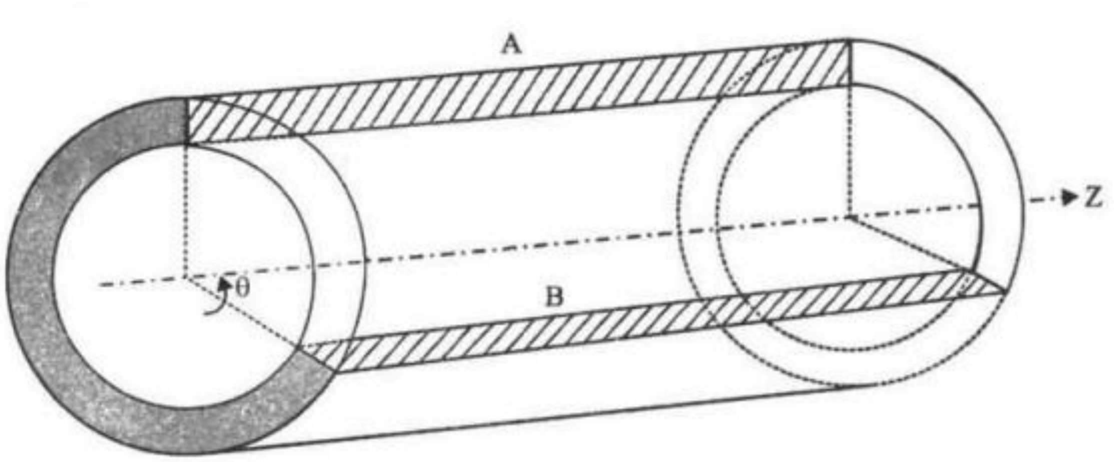
$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

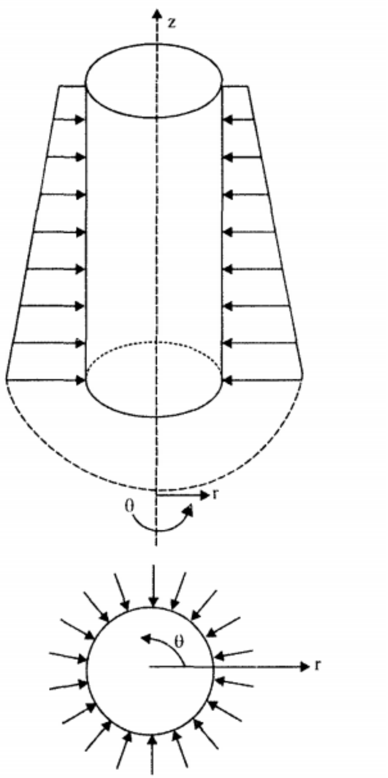
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

- **Axisymmetric element**

When the geometry, boundary condition, load and material properties are identical with respective the axis of symmetry of three dimensional element can be converted into two dimensional axisymmetric problem

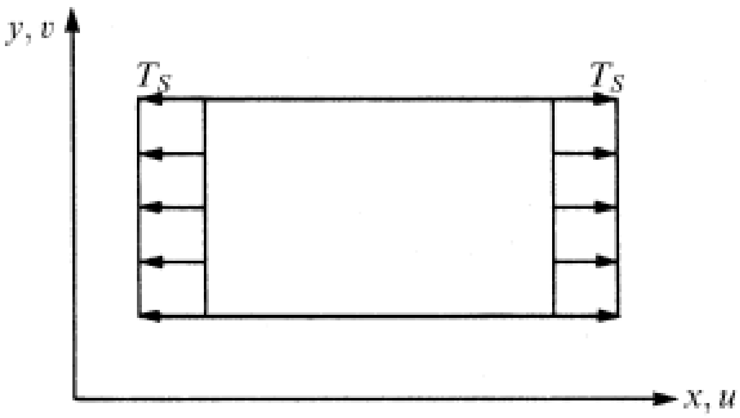




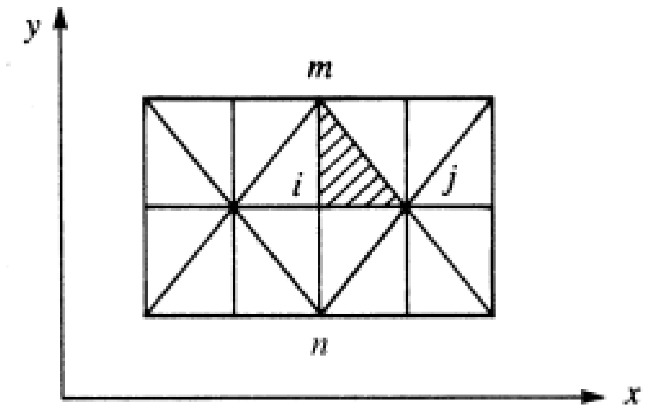
$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \end{Bmatrix} \quad \text{or} \quad \{\sigma\} = [D]\{\varepsilon\}$$

Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations

To illustrate the steps and introduce the basic equations necessary for the plane triangular element, consider the thin plate subjected to tensile surface traction loads TS in Figure

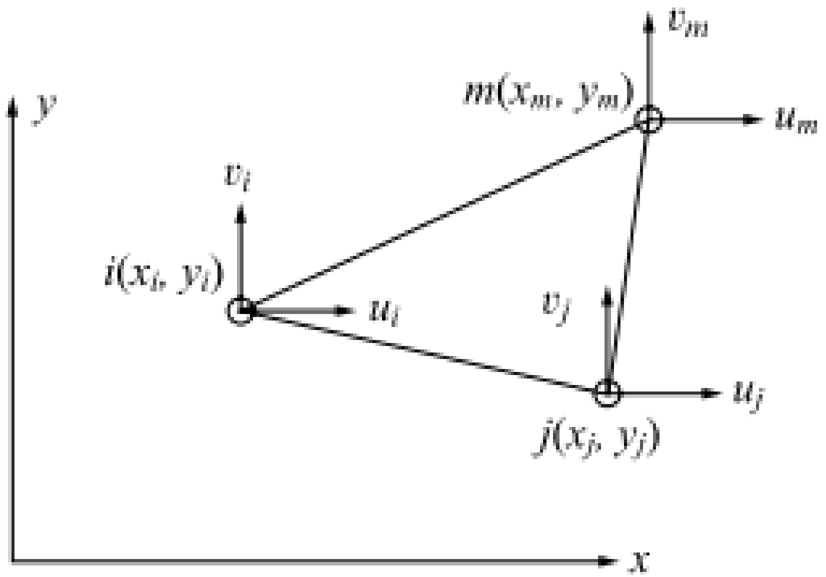


Thin plate in tension



Discretized plate of using triangular elements

- **The discretized plate has been divided into triangular elements, each with nodes such as i; j, and m.**
- **Each node has two degrees of freedom—an x and a y displacement. We will let ui and vi represent the node i displacement components in the x and y directions, respectively.**



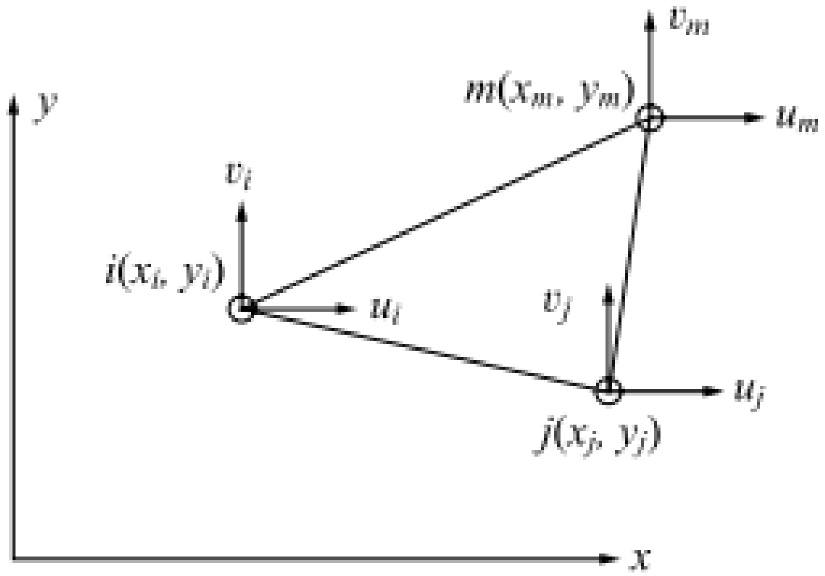
$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

the general displacement function

$$u(x, y) = a_1 + a_2 x + a_3 y$$
$$v(x, y) = a_4 + a_5 x + a_6 y \qquad 1$$

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \qquad 2$$

- **To obtain the a's in Eqs. 1 we begin by substituting the coordinates of the nodal points into Eqs. 1 to yield**



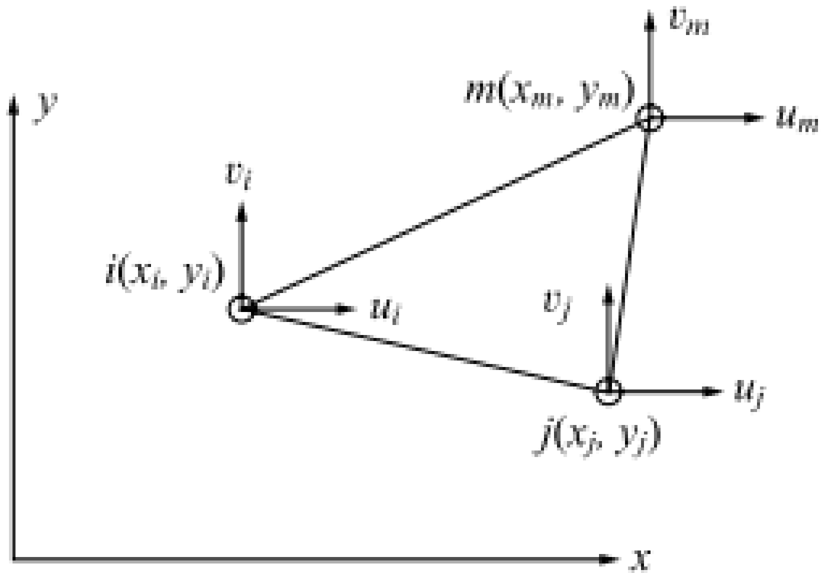
$$u_i = u(x_i, y_i) = a_1 + a_2 x_i + a_3 y_i$$
$$u_j = u(x_j, y_j) = a_1 + a_2 x_j + a_3 y_j$$
$$u_m = u(x_m, y_m) = a_1 + a_2 x_m + a_3 y_m$$
$$v_i = v(x_i, y_i) = a_4 + a_5 x_i + a_6 y_i$$
$$v_j = v(x_j, y_j) = a_4 + a_5 x_j + a_6 y_j$$
$$v_m = v(x_m, y_m) = a_4 + a_5 x_m + a_6 y_m$$

We can solve for the a's beginning with the first three expressed in matrix form as

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{a\} = [x]^{-1}\{u\}$$

The method of cofactors is one possible method for finding the inverse of [x] . Thus



$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

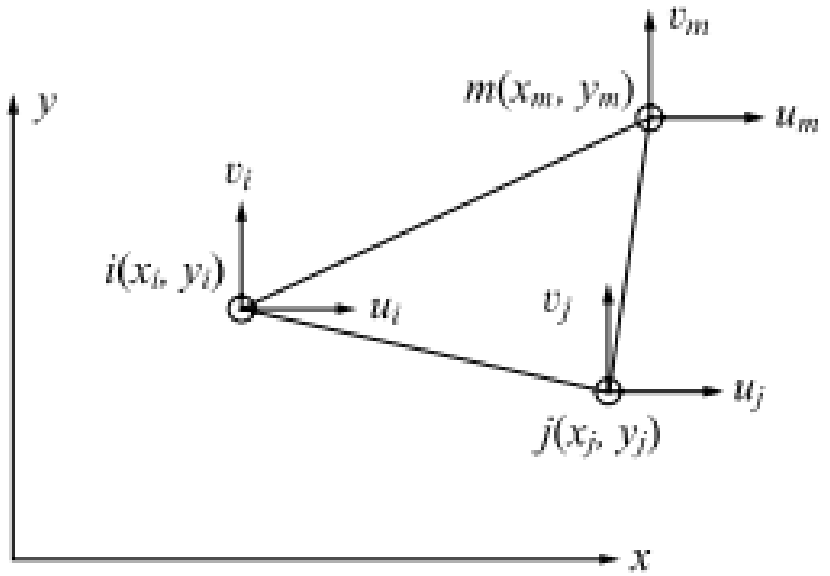
$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

We can solve for the a’s last three expression

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

We will derive the general x displacement function $u(x, y)$ of $\{\psi\}$ (v will follow analogously) in terms of the coordinate variables x and y, known coordinate variables

$\alpha_i, \alpha_j, \ldots, \gamma_m$, and unknown nodal displacements u_i, u_j, and u_n



$$\{u\} = [1 \quad x \quad y]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

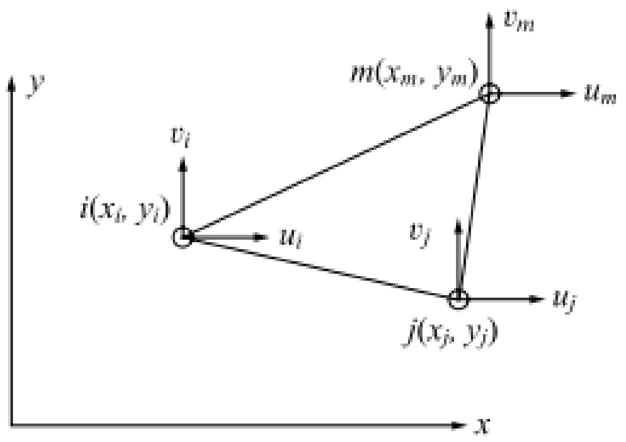
$$u(x, y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\}$$

$$v(x, y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$



$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Define the Strain/Displacement and Stress/Strain Relationships

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m)$$

$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m)$$

Jacobian matrix use to transfer local displacement to global displacement

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m)$$

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\varepsilon\} = [\underline{B}_i \quad \underline{B}_j \quad \underline{B}_m]\begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D][B]\{d\}$$

Using the principle of minimum potential energy, we can generate the equations for a typical constant-strain triangular element. Keep in mind that for the basic plane stress element, the total potential energy is now a function of the nodal displacements,

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s$$

$$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T\{\sigma\}\,dV$$

strain energy

$$\Omega_b = -\iiint_V \{\psi\}^T\{X\}\,dV$$

body forces is given by

$$\Omega_p = -\{d\}^T\{P\}$$

concentrated loads

$$\Omega_s = -\iint_S \{\psi_S\}^T\{T_S\}\,dS$$

distributed loads

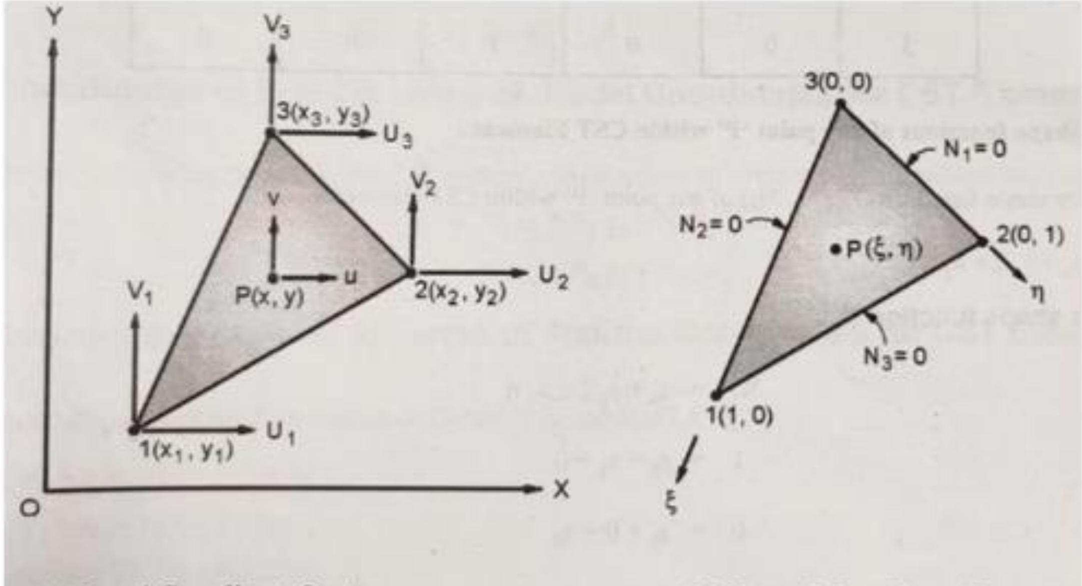
$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{16} \\ k_{21} & k_{22} & \ldots & k_{26} \\ \vdots & \vdots & & \vdots \\ k_{61} & k_{62} & \ldots & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[k_{ii}] = [B_i]^T[D][B_i]tA$$

$$[k_{ij}] = [B_i]^T[D][B_j]tA$$

$$[k_{im}] = [B_i]^T[D][B_m]tA$$

Local & Natural Co-ordinates For CST Element



(a) Local Coordinate System (b) Natural Coordinate System

Node	Shape Functions			Natural Coordinates	
	N_1	N_2	N_3	ξ	η
1	1	0	0	1	0
2	0	1	0	0	1
3	0	0	1	0	0

Three shape functions of any point 'P' within element :

$$N_1 = \xi$$

$$N_2 = \eta$$

$$N_3 = 1 - \xi - \eta$$

Examples on CST

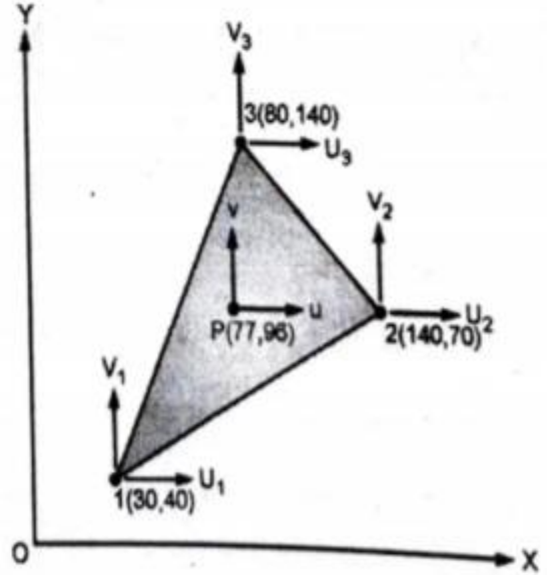
In a triangular element, the nodes 1, 2, and 3 have cartesian coordinates : (30, 40), (140,70), and (80,140) respectively. The displacements, in mm, at nodes 1, 2, and 3 are : (0.1, 0.5), (0.6, 0.5) and (0.4, 0.3) respectively. The point P within the element has cartesian coordinates (77, 96). For point P, determine : (i) the natural coordinates; (ii) the shape functions; and (iii) the displacements.

$1(x_1, y_1) \equiv 1(30, 40)$; $2(x_2, y_2) \equiv 2(140, 70)$; $3(x_3, y_3) \equiv 3(80, 140)$;

$(U_1, V_1) \equiv (0.1, 0.5)$; $(U_2, V_2) \equiv (0.6, 0.5)$; $(U_3, V_3) \equiv (0.4, 0.3)$;

$P(x, y) \equiv P(77, 96)$.

1. Natural Coordinates :



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$x = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3$$

$$y = \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3$$

$$x = (x_1 - x_3)\,\xi + (x_2 - x_3)\,\eta + x_3$$

$$y = (y_1 - y_3)\,\xi + (y_2 - y_3)\,\eta + y_3$$

$$77 = (30 - 80)\,\xi + (140 - 80)\,\eta + 80$$

$$96 = (40 - 140)\,\xi + (70 - 140)\,\eta + 140$$

$$\therefore \quad -50\xi + 60\eta = -3$$

$$-100\xi - 70\eta = -44$$

$$-50\xi + 60\eta = -3$$

$$50\xi + 35\eta = 22$$

$$95\eta = 19$$

3. Displacements of point P :

$$\eta = 0.2$$

$$u = N_1 U_1 + N_2 U_2 + N_3 U_3 = 0.3 \times 0.1 + 0.2 \times 0.6 + 0.5 \times$$

$$50\xi + 35 \times 0.2 = 22$$

$$u = 0.35 \text{ mm}$$

$$\xi = 0.3$$

$$\boldsymbol{\xi = 0.3 \quad \text{and} \quad \eta = 0.2}$$

2. **Shape functions :**

$$N_1 = \xi = 0.3$$

$$N_2 = \eta = 0.2$$

$$N_3 = 1 - \xi - \eta = 1 - 0.3 - 0.2 = 0.5$$

$$\boldsymbol{N_1 = 0.3, \quad N_2 = 0.2 \quad \text{and} \quad N_3 = 0.5}$$

11

3. **Displacements of point P :**

$$u = N_1 U_1 + N_2 U_2 + N_3 U_3 = 0.3 \times 0.1 + 0.2 \times 0.6 + 0.5 \times 0.4$$

$$\text{or} \quad u = 0.35 \text{ mm}$$

$$\text{and} \quad v = N_1 V_1 + N_2 V_2 + N_3 V_3 = 0.3 \times 0.5 + 0.2 \times 0.5 + 0.5 \times 0.3$$

$$\text{or} \quad v = 0.4 \text{ mm}$$

$$\mathbf{u = 0.35\ mm} \quad \text{and} \quad \mathbf{v = 0.4\ mm}$$

The CST element is defined by three nodes located at (1, 1), (4, 2) and (3, 5). For a point P located inside the element, the shape functions N_1 and N_2 are 0.15 and 0.25, respectively. Determine the X and Y coordinates of point P.

Point 1 $(x_1, y_1) \equiv (1, 1)$; Point 2 $(x_2, y_2) \equiv (4, 2)$; Point 3 $(x_3, y_3) \equiv (3, 5)$;

$N_1 = 0.15$; $N_2 = 0.25$.

1. Shape functions :

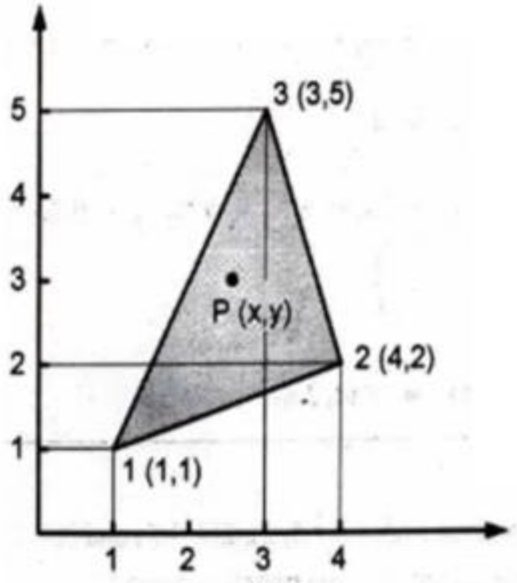


Fig. P. 3.7.2

$$N_3 = 1 - N_1 - N_2 = 1 - 0.15 - 0.25 = 0.6$$

$$N_1 = 0.15, \quad N_2 = 0.25 \quad \text{and} \quad N_3 = 0.6$$

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2. Cartesian coordinates of point P :

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 = 0.15 \times 1 + 0.25 \times 4 + 0.6 \times 3$$

$$\text{or} \quad x = 2.95$$

$$\text{and} \quad y = N_1 y_1 + N_2 y_2 + N_3 y_3 = 0.15 \times 1 + 0.25 \times 2 + 0.6 \times 5$$

$$\text{or} \quad y = 3.65$$

$$P(x, y) = P(2.95, 3.65)$$

The temperatures, in degree Celsius, at nodes 1, 2 and 3 are : 100, 200 and 300 respectively. The coordinates of nodes and that of point P are given below :

Point	x-coordinate	y-coordinate
Node 1	0	0
Node 2	10	0
Node 3	5	8
Point P	5	6

Estimate the shape functions and temperature for point P.

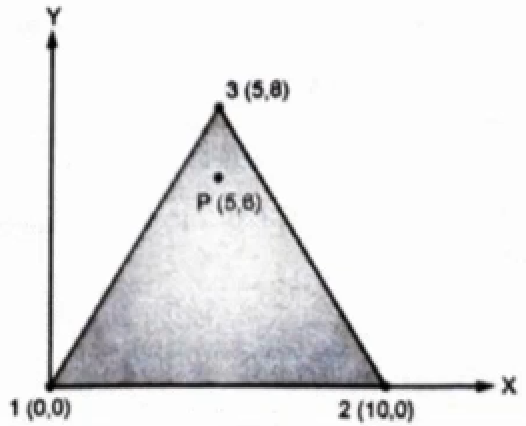
Solution :

Given : $t_1 = 100^\circ C$; $t_2 = 200^\circ C$; $t_3 = 300^\circ C$;

$(x_1, y_1) = (0, 0)$; $(x_2, y_2) = (10, 0)$; $(x_3, y_3) = (5, 8)$;

$P(x, y) = (5, 6)$.

1. Natural coordinates :



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

and $$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$\therefore$ $$x = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3$$

and $$y = \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3$$

$\therefore$ $$x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3$$

and $$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$$

$\therefore$ $$5 = (0 - 5)\xi + (10 - 5)\eta + 5$$

and $$6 = (0 - 8)\xi + (0 - 8)\eta + 8$$

$\therefore$ $$-5\xi + 5\eta = 0$$

and $$-8\xi - 8\eta = -2$$

or $$-\xi + \eta = 0$$

and $$\xi + \eta = \frac{1}{4}$$

$$2\eta = \frac{1}{4}$$

$$\therefore \quad \eta = 0.125$$

$$-\xi + \eta = 0$$

$$\therefore \quad \xi = \eta$$

or $\quad \xi = 0.125$

$$\boldsymbol{\xi = 0.125 \quad \text{and} \quad \eta = 0.125}$$

2. Shape functions :

$$N_1 = \xi = 0.125$$

$$N_2 = \eta = 0.125$$

$$N_3 = 1 - \xi - \eta = 1 - 0.125 - 0.125 = 0.75$$

$$\boldsymbol{N_1 = 0.125, \quad N_2 = 0.125 \text{ and } N_3 = 0.75}$$

3. Temperature at point P :

$$t = N_1 t_1 + N_2 t_2 + N_3 t_3$$

$$= 0.125 \times 100 + 0.125 \times 200 + 0.75 \times 300$$

or $\quad \boldsymbol{t = 262.5^\circ C}$

A rectangular plate of size is 75 mm × 50 mm × 12.5 mm subjected to inplane load of 4500N, as shown in Fig. P. 3.7.11(a). The modulus of elasticity and Poisson's ratio for plate material are 200 × 10^3 N/mm^2 and 0.25 respectively. Model the plate with two CST elements and determine :

(i) the global stiffness matrix ;

(ii) the nodal displacements ;

(iii) the reaction forces at the supports; and

(iv) the stresses in each element.

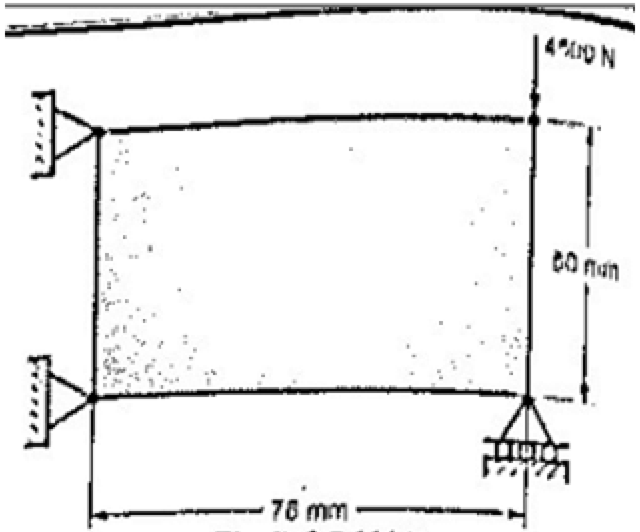


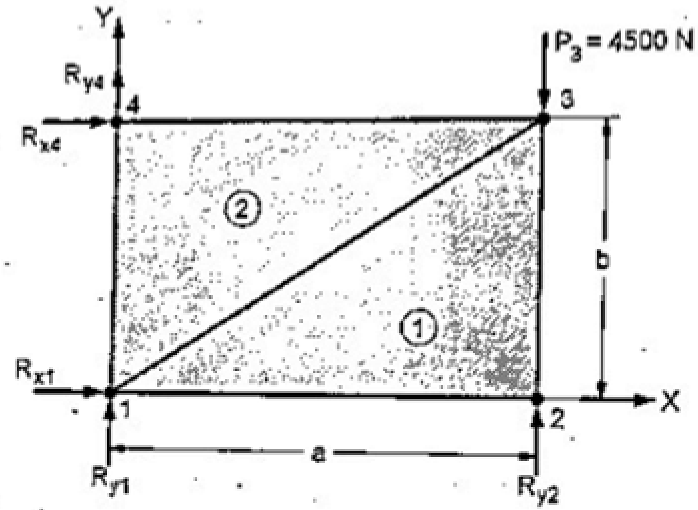
Fig. P. 3.7.11(a)

Solution :

Given : a = 75 mm ; b = 50 mm ; t = 12.5 mm ;

P_3 = 4500 N ; E = 200 × 10^3 N/mm^2 ; ν = 0.25.

1. Discretization :



- The element connectivity table for the assembly is given in Table P. 3.7.11(a).

Table P. 3.7.11(a): Element Connectivity

Element Number ⓔ	Global node Number 'n' of		
	Local Node 1	Local Node 2	Local Node 3
①	1	2	3
②	1	3	4

- The total d.o.f. of assembly, N = D.O.F per node Number of nodes in assembly = 2 × 4 = 8 N
- The dimension of the global stiffness matrix [k] = (8 × 8)
- The dimension of the global load vector {f} = (8 × 1)

The dimension of the global nodal displacement vector, $\{u_N\} = (8 \times 1)$

The element stress-strain matrix for plane stress condition is given by,

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = \frac{E}{[1-(0.25)^2]}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{(1-0.25)}{2} \end{bmatrix}$$

$$\text{or } [D] = 1.0667\ E\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Element 1 :

The coordinates of local nodes of element 1 are given in Table P. 3.7.11(b).

Table P. 3.7.11(b): Coordinates of Nodes of Element 1

Local Node	Coordinates	
	x	y
1	$x_1 = 0$	$y_1 = 0$
2	$x_2 = a$	$y_2 = 0$
3	$x_3 = a$	$y_3 = b$

$$|J|_1 = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix} = \begin{vmatrix} (x_1-x_3) & (y_1-y_3) \\ (x_2-x_3) & (y_2-y_3) \end{vmatrix}$$

$$= \begin{vmatrix} (0-a) & (0-b) \\ (a-a) & (0-b) \end{vmatrix} = \begin{vmatrix} -a & -b \\ 0 & -b \end{vmatrix}$$

$$\text{or} \qquad |J|_1 = ab$$

or $\quad |J|_1 = ab$

nt strain-nodal displacement matrix (element 2) :

The element strain-nodal displacement matrix is given by,

$$[B]_1 = \frac{1}{|J|_1}\begin{bmatrix} y_{23} & 0 & -y_{13} & 0 & y_{12} & 0 \\ 0 & -x_{23} & 0 & x_{13} & 0 & -x_{12} \\ -x_{23} & y_{23} & x_{13} & -y_{13} & -x_{12} & y_{12} \end{bmatrix}$$

$$= \frac{1}{|J|_1}\begin{bmatrix} (y_2-y_3) & 0 & -(y_1-y_3) & 0 & (y_1-y_2) & 0 \\ 0 & -(x_2-x_3) & 0 & (x_1-x_3) & 0 & -(x_1-x_2) \\ -(x_2-x_3) & (y_2-y_3) & (x_1-x_3) & -(y_1-y_3) & -(x_1-x_2) & (y_1-y_2) \end{bmatrix}$$

$$= \frac{1}{ab}\begin{bmatrix} (0-b) & 0 & -(0-b) & 0 & (0-0) & 0 \\ 0 & -(a-a) & 0 & (0-a) & 0 & -(0-a) \\ -(a-a) & (0-b) & (0-a) & -(0-b) & -(0-a) & (0-0) \end{bmatrix}$$

or $$[B]_1 = \frac{1}{ab}\begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$[B]_1^T = \frac{1}{ab}\begin{bmatrix} -b & 0 & 0 \\ 0 & 0 & -b \\ b & 0 & -a \\ 0 & -a & b \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix}$$

- **Element stiffness matrix (element 1) :**

$$A_1 = \frac{1}{2}\,(\text{Magnitude of } |J|) = \frac{ab}{2}$$

The element stiffness matrix is given by,

$$[k]_1 = t\,A_1\,[B]_1^T\,[D]\,[B]_1$$

$$= t \times \frac{ab}{2} \times \frac{1}{ab}\begin{bmatrix} -b & 0 & 0 \\ 0 & 0 & -b \\ b & 0 & -a \\ 0 & -a & b \\ 0 & 0 & a \\ 0 & a & 0 \end{bmatrix} \times 1.0667\,E \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \frac{1}{ab}\begin{bmatrix} -b & 0 & b & 0 \\ 0 & 0 & 0 & -a \\ 0 & -b & -a & b \end{bmatrix}$$

$$= \frac{0.5333\,t\,E}{ab}\begin{bmatrix} -b & -0.25b & 0 \\ 0 & 0 & -0.375b \\ b & 0.25b & -0.375a \\ -0.25\,a & -a & 0.375b \\ 0 & 0 & 0.375a \\ 0.25a & a & 0 \end{bmatrix} \times \begin{bmatrix} -b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & a \\ 0 & -b & -a & b & a & 0 \end{bmatrix}$$

$$\text{or}\quad [k_I] = \frac{0.5333tE}{ab}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} b^2 & 0 & -b^2 & 0.25\,ab & 0 & -0.25\,ab \\ 0 & 0.375b^2 & 0.375\,ab & -0.375\,b^2 & -0.375\,ab & 0 \\ -b^2 & 0.375\,ab & (b^2+0.375\,a^2) & -0.625\,ab & -0.375\,a^2 & 0.25ab \\ 0.25\,ab & -0.375\,b^2 & -0.625\,ab & (a^2+0.375\,b^2) & 0.375\,ab & -a^2 \\ 0 & -0.375\,ab & -0.375\,a^2 & 0.375\,ab & 0.375\,a^2 & 0 \\ -0.25\,ab & 0 & 0.25\,ab & -a^2 & 0 & a^2 \end{bmatrix} \end{array}\begin{array}{c} n \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\quad \text{N/mm}$$

4. Element 2 :

The coordinates of local nodes of element 2 are given in Table P. 3.7.11(c).

Table P. 3.7.11(c): Coordinates of Nodes of Element 2

Local Node	Coordinates	
	x	y
1	$x_1 = 0$	$y_1 = 0$
2	$x_2 = a$	$y_2 = b$
3	$x_3 = 0$	$y_3 = b$

$$|J|_2 = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix} = \begin{vmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{vmatrix}$$

$$= \begin{vmatrix} [illegible] & (0-b) \\ (a-0) & (b-b) \end{vmatrix} = \begin{vmatrix} 0 & -b \\ a & 0 \end{vmatrix}$$

$$\text{or} \qquad |J|_2 = ab$$

Element strain-nodal displacement matrix (element 2) :

The element strain-nodal displacement matrix is given by,

$$[B]_2 = \frac{1}{|J|_2}\begin{bmatrix} y_{23} & 0 & -y_{13} & 0 & y_{12} & 0 \\ 0 & -x_{23} & 0 & x_{13} & 0 & -x_{12} \\ -x_{23} & y_{23} & x_{13} & -y_{13} & -x_{12} & y_{12} \end{bmatrix}$$

$$= \frac{1}{|J|_2}\begin{bmatrix} (y_2-y_3) & 0 & -(y_1-y_3) & 0 & (y_1-y_2) & 0 \\ 0 & -(x_2-x_3) & 0 & (x_1-x_3) & 0 & -(x_1-x_2) \\ -(x_2-x_3) & (y_2-y_3) & (x_1-x_3) & -(y_1-y_3) & -(x_1-x_2) & (y_1-y_2) \end{bmatrix}$$

$$= \frac{1}{ab}\begin{bmatrix} (b-b) & 0 & -(0-b) & 0 & (0-b) & 0 \\ 0 & -(a-0) & 0 & (0-0) & 0 & -(0-a) \\ -(a-0) & (b-b) & (0-0) & -(0-b) & -(0-a) & (0-b) \end{bmatrix}$$

$$[B]_2 = \frac{1}{ab}\begin{bmatrix} 0 & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & 0 & 0 & b & a & -b \end{bmatrix}$$

$$[B]_2^T = \frac{1}{ab}\begin{bmatrix} 0 & 0 & -a \\ 0 & -a & 0 \\ b & 0 & 0 \\ 0 & 0 & b \\ -b & 0 & a \\ 0 & a & -b \end{bmatrix}$$

- **Element stiffness matrix (element 2) :**

$$A_2 = \frac{1}{2}\,(\text{Magnitude of } |J|_2) = \frac{ab}{2}$$

The element stiffness matrix is given by,

$$[k]_2 = t\,A_2\,[B]_2^T\,[D]\,[B]_2$$

$$= t \times \frac{ab}{2} \times \frac{1}{ab}\begin{bmatrix} 0 & 0 & -a \\ 0 & -a & 0 \\ b & 0 & 0 \\ 0 & 0 & b \\ -b & 0 & a \\ 0 & a & -b \end{bmatrix} \times 1.0667\,E \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \frac{1}{ab}\begin{bmatrix} 0 & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & 0 & 0 & b & a & -b \end{bmatrix}$$

$$= \frac{0.5333\,t\,E}{ab}\begin{bmatrix} 0 & 0 & -0.375a \\ -0.25a & -a & 0 \\ b & 0.25b & 0 \\ 0 & 0 & 0.375b \\ -b & -0.25b & 0.375a \\ 0.25a & a & -0.375b \end{bmatrix} \times \begin{bmatrix} 0 & 0 & b & 0 & -b & 0 \\ 0 & -a & 0 & 0 & 0 & a \\ -a & 0 & 0 & b & a & -b \end{bmatrix}$$

$$
[k]_2 = \frac{0.5333tE}{ab}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
0.375a^2 & 0 & 0 & -0.375ab & -0.375a^2 & 0.375ab \\
0 & a^2 & -0.25ab & 0 & 0.25ab & -a^2 \\
0 & -0.25ab & b^2 & 0 & -b^2 & 0.25ab \\
-0.375ab & 0 & 0 & 0.375b^2 & 0.375ab & -0.375b^2 \\
-0.375a^2 & 0.25ab & -b^2 & 0.375ab & (b^2+0.375a^2) & -0.625ab \\
0.375ab & -a^2 & 0.25ab & -0.375b^2 & -0.625ab & (a^2+0.375b^2)
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\ \text{N/mm}
$$

Global stiffness matrix :

$$[K] = [k]_1 + [k]_2$$

The global stiffness matrix is obtained by assembling element stiffness matrices $[k]_1$ and $[k]_2$ such that the elements of each stiff rix are placed in appropriate locations in the global stiffness matrix.

Glob dof

$$
\frac{.5333tE}{ab} \times
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
(b^2+0.375a^2) & 0 & -b^2 & 0.25ab & 0 & -0.625ab & -0.375a^2 & 0.375ab \\
0 & (a^2+0.375b^2) & 0.375ab & -0.375b^2 & -0.625ab & 0 & 0.25ab & -a^2 \\
-b^2 & 0.375ab & (b^2+0.375a^2) & -0.625ab & -0.375a^2 & 0.25ab & 0 & 0 \\
0.25ab & -0.375b^2 & -0.625ab & (a^2+0.375b^2) & 0.375ab & -a^2 & 0 & 0 \\
0 & -0.625ab & -0.375a^2 & 0.375ab & (b^2+0.375a^2) & 0 & -b^2 & 0.25ab \\
-0.625ab & 0 & 0.25ab & -a^2 & 0 & (a^2+0.375b^2) & 0.375ab & -0.375b^2 \\
-0.375a^2 & 0.25ab & 0 & 0 & -b^2 & 0.375ab & (b^2+0.375a^2) & -0.625ab \\
0.375ab & -a^2 & 0 & 0 & 0.25ab & -0.375b^2 & -0.625ab & (a^2+0.375b^2)
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

$$[K] = \frac{t\,E}{75 \times 50} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 2458.33 & 0 & -1333.33 & 500 & 0 & -1250 & -1125 & 750 \\ 0 & 3500 & 750 & -500 & -1250 & 0 & 500 & -3000 \\ -1333.33 & 750 & 2458.33 & -1250 & -1125 & 500 & 0 & 0 \\ 500 & -500 & -1250 & 3500 & 750 & -3000 & 0 & 0 \\ 0 & -1250 & -1125 & 750 & 2458.33 & 0 & -1333.33 & 500 \\ -1250 & 0 & 500 & -3000 & 0 & 3500 & 750 & -500 \\ -1125 & 500 & 0 & 0 & -1333.33 & 750 & 2458.33 & -1250 \\ 750 & -3000 & 0 & 0 & 500 & -500 & -1250 & 3500 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \text{ N/mm} \quad \ldots(m)$$

$$\{F\} = \begin{Bmatrix} P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{Bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \; N = \begin{Bmatrix} R_{y1} \\ 0 \\ R_{y2} \\ 0 \\ -4500 \\ R_{x4} \\ R_{y4} \end{Bmatrix} \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \; N$$

Global nodal displacement vector :

$$\{U_N\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \\ U_4 \\ V_4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \; \text{mm}$$

Global stiffness-nodal displacement-load relationship :

Hence, the resultant matrix equation for the assembly is,

$$[K]\{U_N\} = \{F\}$$

$$[K]\{U_N\} = \{F\}$$

$$\frac{200 \times 10^3}{5 \times 50} \times \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 2458.33 & 0 & -1333.33 & 500 & 0 & -1250 & -1125 & 750 \\ 0 & 3500 & 750 & -500 & -1250 & 0 & 500 & -3000 \\ -1333.33 & 750 & 2458.33 & -1250 & -1125 & 500 & 0 & 0 \\ 500 & -500 & -1250 & 3500 & 750 & -3000 & 0 & 0 \\ 0 & -1250 & -1125 & 750 & 2458.33 & 0 & -1333.33 & 500 \\ -1250 & 0 & 500 & -3000 & 0 & 3500 & 750 & -500 \\ -1125 & 500 & 0 & 0 & -1333.33 & 750 & 2458.33 & -1250 \\ 750 & -3000 & 0 & 0 & 500 & -500 & -1250 & 3500 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \left\{ \begin{array}{c} R_{x1} \\ R_{y1} \\ 0 \\ R_{y2} \\ 0 \\ -4500 \\ R_{x4} \end{array} \right\} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

9. **Nodal displacements :**

- At nodes 1 and 4, there are hinge supports, while at node 2 there is roller support. Hence,

$U_1 = 0;\quad V_1 = 0;\quad V_2 = 0;\quad U_4 = 0\quad$ and $\quad V_4 = 0$

- D.O.Fs 1, 2, 4, 7, and 8 are fixed. Hence, using *elimination approach*, first, second, fourth, seventh, and eighth rows and column be eliminated form Equation (p). Therefore, Equation (p) becomes,

$$666.667\begin{bmatrix} 2458.33 & -1125 & 500 \\ -1125 & 2458.33 & 0 \\ 500 & 0 & 3500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4500 \end{Bmatrix}$$

$$\begin{bmatrix} 2458.33 & -1125 & 500 \\ -1125 & 2458.33 & 0 \\ 500 & 0 & 3500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6.75 \end{Bmatrix}$$

- Adding $\frac{1125}{2458.33} \times$ row II to row I,

$$\begin{bmatrix} 1943.5 & 0 & 500 \\ -1125 & 2458.33 & 0 \\ 500 & 0 & 3500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6.75 \end{Bmatrix}$$

Adding $\frac{1125}{1943.5} \times$ row I to row II,

$$\begin{bmatrix} 1943.5 & 0 & 500 \\ 0 & 2458.33 & 289.43 \\ 500 & 0 & 3500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6.75 \end{Bmatrix}$$

Subtracting $\frac{500}{1943.5} \times$ row I to row III,

$$\begin{bmatrix} 1943.5 & 0 & 500 \\ 0 & 2458.33 & 289.43 \\ 0 & 0 & 3371.366 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6.75 \end{Bmatrix}$$

- Subtracting $\frac{500}{1943.5} \times$ row I to row III,

$$\begin{bmatrix} 1943.5 & 0 & 500 \\ 0 & 2458.33 & 289.43 \\ 0 & 0 & 3371.366 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6.75 \end{Bmatrix}$$

From above matrix Equation (s),

$$1943.5\, U_2 + 500\, V_3 = 0$$
$$2458.33\, U_3 + 289.43\, V_3 = 0$$
$$3371.366\, V_3 = -6.75$$

From Equation (v),

$$V_3 = -2.002 \times 10^{-3} \text{ mm}$$

Substituting Equation (u) in Equations (t) and (u), we get,

$\therefore \quad U_2 = 0.515 \times 10^{-3}$ mm; $\quad U_3 = 0.2357 \times 10^{-3}$ mm $\quad$ and $\quad V_3 = -2.002 \times 10^{-3}$ mm

Reaction forces at supports ;

From Equation (p),

Example 3.7.1 :

In a triangular element, the nodes 1, 2, and 3 have cartesian coordinates : (30, 40), (140,70), and (80,140) respectively. The displacements, in mm, at nodes 1, 2, and 3 are : (0.1, 0.5), (0.6, 0.5) and (0.4, 0.3) respectively. The point P within the element has cartesian coordinates (77, 96). For point P, determine : (i) the natural coordinates; (ii) the shape functions; and (iii) the displacements.

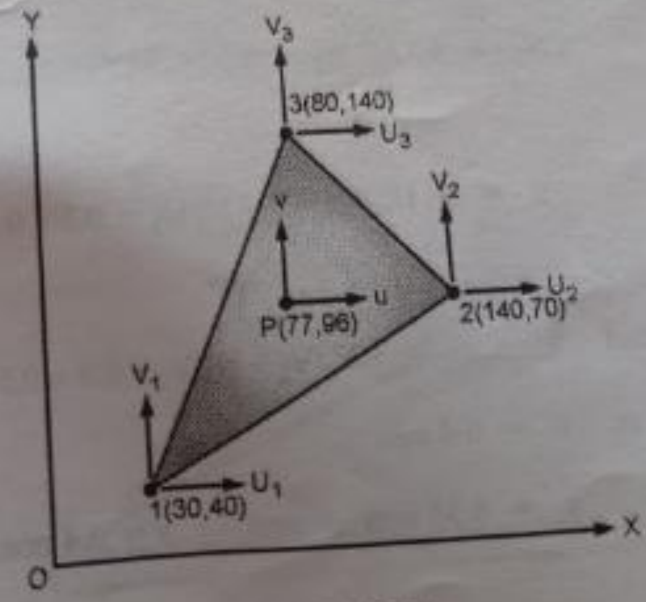
Solution :

Given : $1(x_1, y_1) = 1(30, 40)$; $2(x_2, y_2) = 2(140, 70)$; $3(x_3, y_3) = 3(80, 140)$;

$(U_1, V_1) = (0.1, 0.5)$; $(U_2, V_2) = (0.6, 0.5)$; $(U_3, V_3) = (0.4, 0.3)$;

$P(x, y) = P(77, 96)$.

1. **Natural Coordinates :**



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

and $$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$\therefore \quad x = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3$$

and $$y = \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3$$

$$\therefore \quad x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3$$

and $$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$$

$$\therefore \quad 77 = (30 - 80)\xi + (140 - 80)\eta + 80$$

and $$96 = (40 - 140)\xi + (70 - 140)\eta + 140$$

$$\therefore \quad -50\xi + 60\eta = -3$$

and $$-100\xi - 70\eta = -44$$

or $$-50\xi + 60\eta = -3$$

and $$50\xi + 35\eta = 22$$

Adding Equations (a) and (b)

$$95\eta = 19$$

$$\therefore \quad \eta = 0.2$$

- Adding Equations (a) and (b)

$$95\eta = 19$$

$$\therefore \quad \eta = 0.2$$

- Substituting Equation (c) in Equation (b),

$$50\xi + 35 \times 0.2 = 22$$

$$\xi = 0.3$$

$$\xi = 0.3 \quad \text{and} \quad \eta = 0.2$$

2. **Shape functions :**

$$N_1 = \xi = 0.3$$

$$N_2 = \eta = 0.2$$

$$N_3 = 1 - \xi - \eta = 1 - 0.3 - 0.2 = 0.5$$

$$N_1 = 0.3, \quad N_2 = 0.2 \quad \text{and} \quad N_3 = 0.5$$

3. **Displacements of point P :**

$$u = N_1 U_1 + N_2 U_2 + N_3 U_3 = 0.3 \times 0.1 + 0.2 \times 0.6 + 0.5 \times 0.4$$

or $$u = 0.35 \text{ mm}$$

and $$v = N_1 V_1 + N_2 V_2 + N_3 V_3 = 0.3 \times 0.5 + 0.2 \times 0.3 + 0.5 \times 0.3$$

or $$v = 0.4 \text{ mm}$$

$$u = 0.35 \text{ mm} \quad \text{and} \quad v = 0.4 \text{ mm}$$

Example 3.7.7 :

A CST element is defined by nodes at I (24, 30), J (60, 20), and K (90,50) while the stresses at these nodes are [illegible] MPa respectively. Determine the stress at point P (60, 35).

Solution :

Given : $I(x_1, y_1) = (24, 30)$; $J(x_2, y_2) = (60, 20)$; $K(x_3, y_3) = (90, 50)$; $P(P_x, P_y) = (60, 35)$;

$\sigma_1 = [illegible]\ N/mm^2$; $\sigma_2 = 120\ N/mm^2$; $\sigma_3 = 140\ N/mm^2$.

1. Natural coordinates :

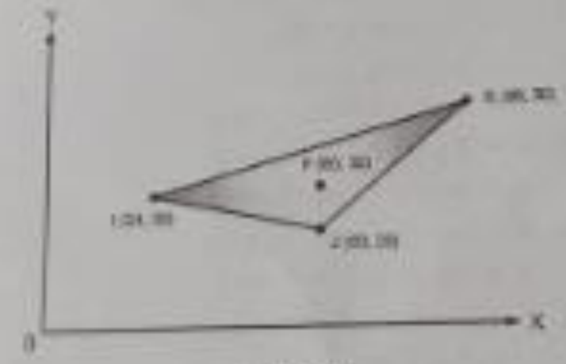


Fig. P. 3.7.7

Refer Fig. P. 3.7.7 ;

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

and $$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$\therefore \quad x = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3$$

and $y = \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3$

$\therefore$ $x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3$

and $y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$

$$60 = (24 - 90)\xi + (60 - 90)\eta + 90$$

and $30 = (30 - 50)\xi + (20 - 50)\eta + 50$

$\therefore$ $66\xi + 30\eta = 30$

$$20\xi + 30\eta = 20$$

Subtracting Equation (b) from Equation (a),

$$46\xi = 10$$

$$\therefore \xi = 0.21$$

Again, from Equation (b),

$$20 \times 0.21 + 30\eta = 20$$

$$\therefore \eta = 0.52$$

$$\xi = 0.21 \quad \text{and} \quad \eta = 0.52$$

2. Shape Function :

$$N_1 = \xi = 0.21$$

$$N_2 = \eta = 0.52$$

$$N_3 = 1 - \xi - \eta = 1 - 0.21 - 0.52 = 0.27$$

$$N_1 = 0.21, \; N_2 = 0.52 \text{ and } N_3 = 0.27$$

3. Stress at point 'P' :

$$\sigma_p = N_1\sigma_I + N_2\sigma_J + N_3\sigma_K$$

$$= 0.21 \times 90 + 0.52 \times 120 + 0.27 \times 160 = 124.5 \text{ N/mm}$$

$\therefore$ **Stress at point 'P' $= 124.5$ N/mm^2**

Thank You

For Your Attention